OPTIMAL NUMBER OF TRIALS FOR MONTE CARLO SIMULATION

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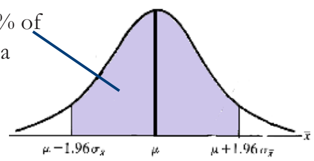
This article presents a way to estimate the number of trials required for a desired confidence interval in the context of a Monte Carlo simulation.

Traditional valuation approaches such as Option Pricing Method ("OPM") or Probability-Weighted Expected Return Method ("PWERM") may not be adequate in providing fair value estimation for financial instruments that require distribution assumptions on multiple input parameters. In such cases, a numerical method, Monte Carlo simulation for instance, is often used. The Monte Carlo simulation is a computerized algorithmic procedure that outputs a wide range of values – typically unknown probability distribution – by simulating one or multiple input parameters via known probability distributions.

This technique is often used to find fair value for financial instruments for which probabilistic distributions are unknown. The simulation procedure is typically repeated multiple times and the average of the results is taken. The theory of law of large numbers requires a simulation to be repeated many times in order to have an accurate estimate. The level of precision, in the context of simulation, is often measured by *confidence interval*: a smaller confidence interval indicates a more robust value estimate and vice versa. In most cases we could have a very good value estimate if a simulation is iterated for anywhere between 100,000 to 500,000 times.

Depending on the complexity of the simulation algorithm and the software used to run the program, even 100K iterations could take several hours. It would be beneficial to know what level of precision, or confidence interval, we could achieve for a certain number of iterations in advance of running the program. Without the confidence interval, the time commitment for a trial-and-error process would be significant. In this article, we will present a simple method to estimate the number of simulations needed for a desired level of precision when running a Monte Carlo simulation.



Statistically, 95% of the area under a normal distribution curve is described as being plus or minus 1.96 standard deviations from the mean. For 90%, the z-statistic is 1.64.

To facilitate further discussion, we will define some terminologies that will be referenced later on.

Confidence level: A percentage used to characterize a confidence interval. The common levels are 95% and 99%.

Level of precision: The maximum degree the true population mean can deviate from the sample mean estimation, subject to a given confidence level. By the symmetric construction of confidence interval, it is the width of a confidence interval.

Confidence interval: Characterized by two parameters: *confidence level* and its associated *-statistic*. For instance, a confidence interval of an estimated population mean is often presented in terms of a percentage, such as 95%. The z –statistic is the standard deviation from the mean. The interpretation of this confidence interval is: we are 95% sure the confidence interval contains the true population mean, which is the subject of estimation.

CALCULATING CONFIDENCE INTERVAL

Central Limit Theorem is at the heart of a confidence interval. The theorem is as follows:

Let $X_1, X_2, ..., X_n$ be a sequence of independent identically distributed random variables with finite mean μ and variance σ^2.

Let $\bar{X} = \sum_{i=1}^{n} X_i \Big/ n$ *be the average of the sample. Then the C.D.F* $F_n(x)$ *of the random variable* $Z_n = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}}$ *converges to the standard normal C.D.F at all points* x *.*

The Central Limit Theorem is so powerful it enables us to work with any distribution. The Theorem states the mean of a sample is normally distributed regardless of the type of distribution of data from which the samples were taken. However, this holds only if the population is significantly larger than the number of samples. The Confidence interval can be calculated as follows:

$$CI = (\bar{x} - z \times \frac{s}{\sqrt{n}}, \bar{x} + z \times \frac{s}{\sqrt{n}}) \qquad (1)$$

where $\bar{x}$ *is the sample mean,* z *is the statistic associated with a certain confidence interval,* s *is the sample standard deviation and* n *is the sample size. In case of a 95% confidence interval, the* z *statistic equals to 1.96 approximately.*

Let's see an example calculation: Suppose $s_1, s_2, ..., s_n$, $n = 200$ are a sequence of independent and identically distributed (i.i.d.) random variables generated from exponential distribution $\exp(\lambda)$ with λ =2. The sample mean and standard deviation are 2.29 and 2.13 respectively. According to equation (1), the 95% confidence interval is (1.995, 2.585) with the mean of 2.298.

SOLVING FOR THE UNKNOWN — 'n'

Equation (1) is the well-known formula for confidence interval calculation, and it is derived from the confidence interval's pure probabilistic form. In order to understand confidence interval, it is helpful to rewrite equation (1) in its probabilistic form.

First, rewrite equation (1) in terms of population mean μ as the following inequality:

$$\bar{x} - z \times \frac{s}{\sqrt{n}} < \mu < \bar{x} + z \times \frac{s}{\sqrt{n}}$$

Divide the above inequality by $s/\sqrt{n}$:

$$\frac{\bar{x}}{s/\sqrt{n}} - z < \frac{\mu}{s/\sqrt{n}} < \frac{\bar{x}}{s/\sqrt{n}} + z$$

Split the above two inequalities and rearrange:

$$\begin{cases} z > \dfrac{\bar{x} - \mu}{s/\sqrt{n}} \\ z < -\dfrac{\bar{x} - \mu}{s/\sqrt{n}} \end{cases}$$

Note that z is the random normal variable. Let's rewrite z in terms of our level of precision ϕ, and then we obtain the following equation in probabilistic form:

$$P\left(-\frac{\phi}{s/\sqrt{n}} < \frac{\bar{x} - \mu}{s/\sqrt{n}} < \frac{\phi}{s/\sqrt{n}}\right) = confidence\ level \quad (2)$$

where s *is the sample standard deviation,* n *is the sample size,* $\bar{x}$ *sample mean,* μ *population mean, and is* ϕ *the level of precision.*

According to the Central Limit Theorem, the middle term in the above inequality is normally distributed with mean zero and standard deviation one. Another way to look at this is that the distribution of the sample mean is scaled by a factor of $\frac{\sqrt{n}}{s}$ so the distribution becomes standard normal. Suppose we choose a 95% confidence level, then the upper and lower bounds of equation (2) are -1.96 and 1.96 respectively.

At this point, the solution to finding the number of simulations required for a desired confidence interval (level of precision) is straightforward: solving the upper and lower bounds of equation (2) for the only unknown variable *n* – the number of simulations. At first glimpse, there seem to be two unknowns n and s; however, s - the sample deviation can be obtained by running the simulation for a number of times.

Caveats: regarding the distribution of variance

If sample data are normally distributed, then the variance follows Chi Square distribution. The distribution changes as sample size n increases. For non-normal random variables, the distribution of sample variance is unknown. Because of this technical bottleneck, the technique presented in this article only serves the need for estimating the number of trials that is needed to achieve a certain desired level of precision. This method, however, does not produce a theoretical minimum number of the simulations needed. Nevertheless, as the sample size increases, the distribution of sample variance approaches to the normal distribution.

TECHNIQUE ILLUSTRATION: CASE STUDY

Suppose we built a simulation model that values an incentive unit whose payoff is a function of two variables: EBITDA and revenue, and we used triangle distribution for the EBITDA growth simulation and Geometric Brownian Motion for the revenue simulation. For this example, we set the level of precision at $0.01 per share. In order to calculate the required number of trials for this level of precision, the following steps are recommended:

1. Run the simulation 500 times (a number that does not take too much time to run but is large enough for the sample standard deviation to converge reasonably well), and calculate the sample standard deviation. Assuming $s = 1.45$ in this case.

2. Choose the confidence level and select the associated z-statistic. We will use 95% and 1.96 for the z-statistic.

3. Solve the equation $\frac{0.01}{1.45/\sqrt{n}} = 1.96$ for n, which n =80,769.

4. Rerun the simulation for approximately 80,769 times to achieve the desired level of precision.

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